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Polarizations as States and Their Evolution in Geometric Algebra Terms with Variable Complex Plane

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Abstract

Recently suggested scheme [1] of quantum computing uses g-qubit states as circular polarizations from the solution of Maxwell equations in terms of geometric algebra, along with clear definition of a complex plane as bivector in three dimensions. Here all the details of receiving the solution, and its polarization transformations are analyzed. The results can particularly be applied to the problems of quantum computing and quantum cryptography. The suggested formalism replaces conventional quantum mechanics states as objects constructed in complex vector Hilbert space framework by geometrically feasible framework of multivectors.

Keywords

Quantum Mechanics, Quantum Computing, Geometric Algebra, Maxwell Equations

1. Introduction

The circular polarized electromagnetic waves are the only type of waves following from the solution of Maxwell equations in free space done in geometric algebra terms.

Let's take the electromagnetic field in the form:

$$F = F_0 \exp\left[I_S\left(\omega t - \boldsymbol{k}\cdot\boldsymbol{r}\right)\right] \tag{1}$$

requiring that it satisfies the Maxwell system of equations in free space, which in geometrical algebra terms is one equation:

$$\left(\partial_t + \nabla\right)F = 0 \tag{2}$$

where $\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$ and multiplications are the geometrical algebra ones.

Element F_0 in (1) is a constant element of geometric algebra G_3 and I_S is unit value bivector of a plane S in three dimensions, that is a generalization of the imaginary unit [2], [3]. The exponent in (1) is unit value element of G_3^+ [3]:

$$\mathrm{e}^{I_S\varphi} = \cos\varphi + I_S \sin\varphi, \ \ \varphi = \omega t - \boldsymbol{k}\cdot\boldsymbol{r}$$

Solution of (2) should be sum of a vector (electric field $\boldsymbol{E}$) and bivector (magnetic field $I_3\boldsymbol{H}$):

$$F = \boldsymbol{E} + I_3\boldsymbol{H}$$

with some initial conditions:

$$\boldsymbol{E} + I_3\boldsymbol{H}\big|_{t=0,\boldsymbol{r}=0} = F_0 = \boldsymbol{E}\big|_{t=0,\boldsymbol{r}=0} + I_3\boldsymbol{H}\big|_{t=0,\boldsymbol{r}=0} = \boldsymbol{E}_0 + I_3\boldsymbol{H}_0$$

In the magnetic field $I_3\boldsymbol{H}$ the item I_3 is unit pseudoscalar in three dimensions assumed to be the right-hand screw oriented volume, relative to an ordered triple of orthonormal vectors.

Substitution of (1) into the Maxwell's (2) will exactly show us what the solution looks like.

2. Solution in the Geometric Algebra Terms

The derivative by time gives

$$\frac{\partial}{\partial t}F = F_0\mathrm{e}^{I_S\varphi}I_S\frac{\partial}{\partial t}(\omega t - \boldsymbol{k}\cdot\boldsymbol{r}) = F_0\mathrm{e}^{I_S\varphi}I_S\omega = FI_S\omega$$

The geometric algebra product ∇F is:

$$\nabla F = F_0 I_S \mathrm{e}^{I_S\varphi}\nabla(\omega t - \boldsymbol{k}\cdot\boldsymbol{r}) = -F_0\mathrm{e}^{I_S\varphi}I_S\boldsymbol{k} = -FI_S\boldsymbol{k}$$

or

$$\nabla F = F_0\mathrm{e}^{I_S\varphi}\nabla(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})I_S = -F_0\mathrm{e}^{I_S\varphi}\boldsymbol{k}I_S = -F\boldsymbol{k}I_S,$$

depending on do we write $I_S(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})$ or $(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})I_S$. The result should be the same since $\omega t - \boldsymbol{k}\cdot\boldsymbol{r}$ is a scalar.

Commutativity $I_S\boldsymbol{k} = \boldsymbol{k}I_S$ is true only if $\boldsymbol{k}\times I_3I_S = 0$. The following agreement takes place between orientation of I_3, orientation of I_S and direction of vector $\boldsymbol{k}$ [3]. The vector $I_3I_S = I_SI_3$ is orthogonal to the plane of I_S and its direction is defined by orientations of I_3 and I_S. Rotation of right/left hand screw defined by orientation of I_S gives movement of right/left hand screw. This is the direction of the vector $I_3I_S = I_SI_3$. That means that the matching between $\hat{k}$ and I_S should be $\hat{k} = \pm I_3I_S$ or $\hat{k}I_S = \mp I_3$[1].

Assuming that orientation is $I_3 = \hat{k}I_S$, the Maxwell equation becomes:

$$F\left(I_S\omega - I_3|\boldsymbol{k}|\right) = F\left(\omega I_S - |\boldsymbol{k}|\hat{k}I_S\right) = 0$$

or

[1]For any vector we write $\hat{a} = \boldsymbol{a}/|\boldsymbol{a}|$.

$$\left(\boldsymbol{E}+I_3\boldsymbol{H}\right)\omega=\left(\boldsymbol{E}+I_3\boldsymbol{H}\right)\boldsymbol{k}$$

Left hand side is sum of vector and bivector, while right hand side is scalar $\boldsymbol{E}\cdot\boldsymbol{k}$ plus bivector $\boldsymbol{E}\wedge\boldsymbol{k}$, plus pseudoscalar $I_3\left(\boldsymbol{H}\cdot\boldsymbol{k}\right)$, plus vector $I_3\left(\boldsymbol{H}\wedge\boldsymbol{k}\right)$. It follows that both $\boldsymbol{E}$ and $\boldsymbol{H}$ lie on the plane of I_S and then:

$$\omega\boldsymbol{E}=I_3\boldsymbol{Hk},\ \omega I_3\boldsymbol{H}=\boldsymbol{Ek}\rightarrow\frac{\omega^2}{|\boldsymbol{k}|^2}I_3\boldsymbol{Hk}=\omega\boldsymbol{E}$$

Thus, $\omega=|\boldsymbol{k}|$ and we get equation $I_3\boldsymbol{H}\hat{k}=\boldsymbol{E}$ from which particularly follows $|\boldsymbol{E}|^2=|\boldsymbol{H}|^2$ and $\hat{E}\hat{k}\hat{H}=I_3$.

The result for the case $I_3=\hat{k}I_S$ is that the solution of (2) is

$$F=\left(\boldsymbol{E}_0+I_3\boldsymbol{H}_0\right)\exp\left[I_S\left(\omega t-\boldsymbol{k}\cdot\boldsymbol{r}\right)\right]$$

where $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ are arbitrary mutually orthogonal vectors of equal length, lying on the plane *S*. Vector $\boldsymbol{k}$ should be normal to that plane, $\hat{k}=-I_3I_S$ and $|\boldsymbol{k}|=\omega$.

In the above result the sense of the I_S orientation and the direction of $\vec{k}$ were assumed to agree with $I_3=\hat{k}I_S$. Opposite orientation, $-I_3=\hat{k}I_S$, that's $\boldsymbol{k}$ and I_S compose left hand screw and $\hat{k}=I_3I_S$, will give solution $F=F_0\exp\left[I_S\left(\omega t-\boldsymbol{k}\cdot\boldsymbol{r}\right)\right]$ with $\hat{E}\hat{H}\hat{k}=I_3$.

Summary:

For a plane *S* in three dimensions Maxwell equation (2) has two solutions

- $F_+=\left(\boldsymbol{E}_0+I_3\boldsymbol{H}_0\right)\exp\left[I_S\left(\omega t-\boldsymbol{k}_+\cdot\boldsymbol{r}\right)\right]$, with $\hat{k}_+=I_3I_S$, $\hat{E}\hat{H}\hat{k}_+=I_3$, and the triple $\left\{\hat{E},\hat{H},\hat{k}_+\right\}$ is right hand screw oriented, that's rotation of $\hat{E}$ to $\hat{H}$ by π/2 gives movement of *right hand screw* in the direction of $\boldsymbol{k}_+=|\boldsymbol{k}|I_3I_S$.
- $F_-=\left(\boldsymbol{E}_0+I_3\boldsymbol{H}_0\right)\exp\left[I_S\left(\omega t-\boldsymbol{k}_-\cdot\boldsymbol{r}\right)\right]$, with $\hat{k}_-=-I_3I_S$, $\hat{E}\hat{H}\hat{k}_-=-I_3$, and the triple $\left\{\hat{E},\hat{H},\hat{k}_-\right\}$ is left hand screw oriented, that's rotation of $\hat{E}$ to $\hat{H}$ by π/2 gives movement of *left hand screw* in the direction of $\boldsymbol{k}_-=-|\boldsymbol{k}|I_3I_S$ or, equivalently, movement of *right hand screw* in the opposite direction, $-\boldsymbol{k}_-$.
- $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$, initial values of $\boldsymbol{E}$ and $\boldsymbol{H}$, are arbitrary mutually orthogonal vectors of equal length, lying on the plane *S*. Vectors $\boldsymbol{k}_\pm=\pm|\boldsymbol{k}_\pm|I_3I_S$ are normal to that plane. The length of the wave vectors $|\boldsymbol{k}_\pm|$ is equal to angular frequency *ω*.

Maxwell Equation (2) is a linear one. Then any linear combination of F_+ and F_- saving the structure of (1) will also be a solution.

Let's write:

$$\begin{cases}F_+=\left(\boldsymbol{E}_0+I_3\boldsymbol{H}_0\right)\exp\left[I_S\omega t\right]\exp\left[-I_S\left[\left(I_3I_S\right)\cdot\boldsymbol{r}\right]\right]\\F_-=\left(\boldsymbol{E}_0+I_3\boldsymbol{H}_0\right)\exp\left[I_S\omega t\right]\exp\left[I_S\left[\left(I_3I_S\right)\cdot\boldsymbol{r}\right]\right]\end{cases}\tag{3}$$

Then for arbitrary scalars *λ* and *μ*:

$$\lambda F_++\mu F_-=\left(\boldsymbol{E}_0+I_3\boldsymbol{H}_0\right)\mathrm{e}^{I_S\omega t}\left(\lambda\mathrm{e}^{-I_S\left[\left(I_3I_S\right)\cdot\boldsymbol{r}\right]}+\mu\mathrm{e}^{I_S\left[\left(I_3I_S\right)\cdot\boldsymbol{r}\right]}\right)\tag{4}$$

is solution of (2). The item in second parenthesis is weighted linear combination of two states with the same phase in the same plane but opposite sense of orientation. The states are strictly coupled, entangled if you prefer, because bivector plane should be the same for both, does not matter what happens with it.

One another option is:

$$\begin{aligned}&(\lambda_1 + I_3\mu_1)(\boldsymbol{E}_0 + I_3\boldsymbol{H}_0)\exp\left[I_S\omega\left(t-(I_3 I_S)\cdot\boldsymbol{r}\right)\right]\\&+(\lambda_2 + I_3\mu_2)(\boldsymbol{E}_0 + I_3\boldsymbol{H}_0)\exp\left[I_S\omega\left(t+(I_3 I_S)\cdot\boldsymbol{r}\right)\right]\\&=\left[\lambda_1\boldsymbol{E}_0 - \mu_1\boldsymbol{H}_0 + I_3(\mu_1\boldsymbol{E}_0 + \lambda_1\boldsymbol{H}_0)\right]\exp\left[I_S\omega\left(t-(I_3 I_S)\cdot\boldsymbol{r}\right)\right]\\&\quad+\left[\lambda_2\boldsymbol{E}_0 - \mu_2\boldsymbol{H}_0 + I_3(\mu_2\boldsymbol{E}_0 + \lambda_2\boldsymbol{H}_0)\right]\exp\left[I_S\omega\left(t+(I_3 I_S)\cdot\boldsymbol{r}\right)\right]\end{aligned}$$

which is just rotation, along with possible change of length, of electric and magnetic initial vectors in their plane.

3. Transformations of Polarization States

Polarizations, in our approach, exponents in the solution of (3), have the form of states [3], that's elements of G_3^+: $G_3^+ \ni \alpha + I_S\beta \equiv \cos\varphi + I_S\sin\varphi = e^{I_S\varphi}$, distributed in $(t,\boldsymbol{r})$ space. They are operators than can act on observables, also elements of G_3, particularly other polarizations. Such states can be depicted in the current geometric algebra formalism using a triple of basis bivectors in three dimensions $\{B_1, B_2, B_3\}$ (**Figure 1**):

The basis bivectors satisfy multiplication rules (in the righth and screw orientation of I_3):

$$B_1 B_2 = -B_3,\quad B_1 B_3 = B_2,\quad B_2 B_3 = -B_1$$

One can identify basis bivectors with usual coordinate planes: $B_1 = \hat{y}\hat{z}$, $B_2 = \hat{z}\hat{x}$, $B_3 = \hat{x}\hat{y}$. Any one of these three bivectors can be taken as explicitly identifying imaginary unit, though any unit value bivector in three dimensions can take the role [2], [4].

Thus:

$$\alpha + I_S\beta = \alpha + \beta(b_1 B_1 + b_2 B_2 + b_3 B_3) \equiv \alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3$$

The difference between units of information in classical computational scheme, quantum mechanical conventional computations (qubits) and geometric algebra

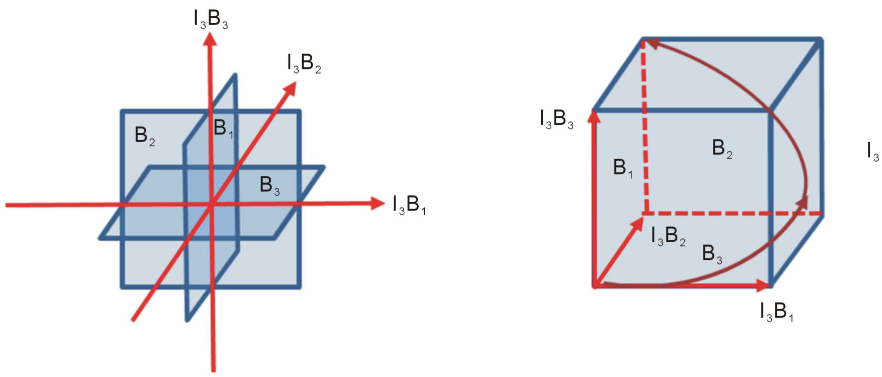

Figure 1. Basis of bivectors and unit value pseudoscalar.

scheme (g-qubits) with variable explicitly defined complex plane is seen from Figure 2.

Circular polarizations received as solutions of Maxwell Equation (2) is an excellent choice to have such g-qubits in a lab.

Commonly accepted idea to use systems of qubits to tremendously increase speed of computations is based on assumption of entanglement – roughly speaking when touching one qubit all the other in the system react instantly, in no time. A bit strange, though you should not care about that because our paradigm is very different.

Assume we have some general state:

$$\alpha + I_S\beta = \alpha + \beta\left(b_1B_1 + b_2B_2 + b_3B_3\right) \equiv \alpha + \beta_1B_1 + \beta_2B_2 + \beta_3B_3$$

The state can be identified as a point $(\alpha_1, \beta_1, \beta_2, \beta_3)$ on unit sphere S^3. It can be subjected to a Clifford translation

$$\alpha + I_S\beta \Rightarrow \mathrm{e}^{I_{Cl}\Delta\psi}\left(\alpha + I_S\beta\right)$$

executing displacement $\Delta\psi$ at point $(\alpha_1, \beta_1, \beta_2, \beta_3)$ along intersection of S^3 with the unit bivector plane I_{Cl}.

Let's make notations more like conventional quantum mechanical ones. I will write:

$$\alpha + I_S\beta \equiv \left|g\right\rangle_{(\alpha,\beta,I_S)},\ \overline{\alpha + I_S\beta} = \alpha - I_S\beta \equiv \left\langle g\right|_{(\alpha,\beta,I_S)}$$

and use Hamiltonian like form of the Clifford translation bivector.

Conventional Hamiltonian

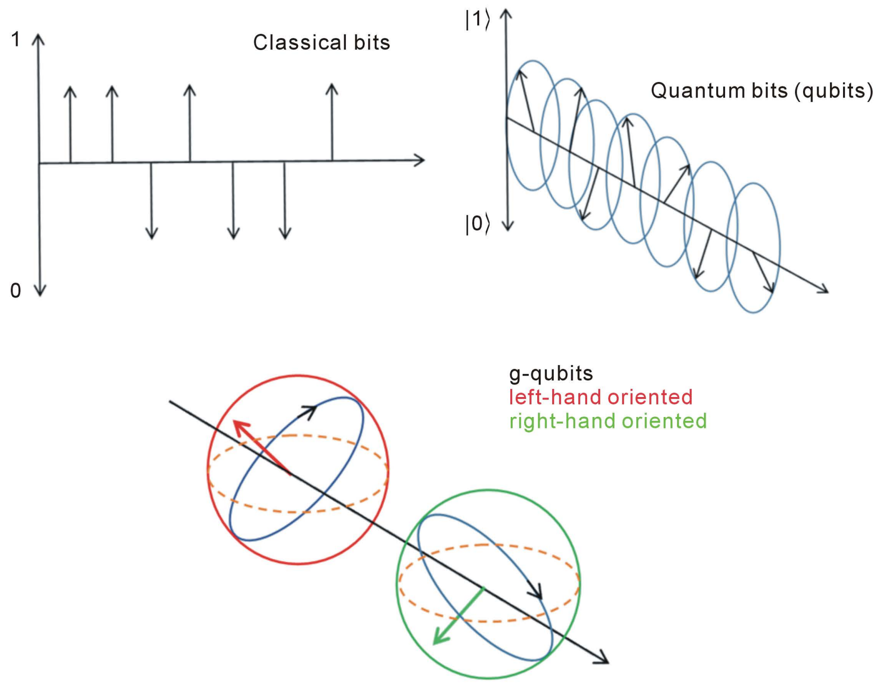

Figure 2. Differences between bits, qubits and g-qubits.

$$\begin{pmatrix} \gamma+\gamma_1 & \gamma_2-i\gamma_3 \\ \gamma_2+i\gamma_3 & \gamma-\gamma_1 \end{pmatrix},$$

with removed not important scalar γ, has the lift in G_3^+ [3]:

$$\mathcal{H}=I_3\left(\gamma_1 B_1+\gamma_2 B_2+\gamma_3 B_3\right)$$

Then the associated Clifford translation plane bivector is $-I_3\mathcal{H}(t)$. By normalizing the bivector to unit value we get generalization of imaginary unit

$$i\Rightarrow I_3\frac{\mathcal{H}(t)}{\left|\mathcal{H}(t)\right|},$$

that is critical for the whole approach. Therefore, for some Δt, Clifford translation for a given Hamiltonian is:

$$\begin{aligned} &\left|g\left(t+\Delta t\right)\right\rangle_{\left(\alpha(t+\Delta t),\beta(t+\Delta t),I_S(t+\Delta t)\right)} \\ &=\mathrm{e}^{-\left(I_3\frac{\mathcal{H}(t)}{|\mathcal{H}(t)|}\right)|\mathcal{H}(t)|\Delta t}\left|g\left(t\right)\right\rangle_{\left(\alpha(t),\beta(t),I_S(t)\right)} \end{aligned} \tag{5}$$

For an arbitrary sequence of infinitesimal Clifford translations, the final state is integral[2]

$$\int \mathrm{e}^{-I_{H(l)}\mathcal{H}(l)\mathrm{d}l}\left|g\left(l\right)\right\rangle_{\left(\alpha(l),\beta(l),I_S(l)\right)}$$

along the curve on unit sphere S^3 composed of infinitesimal displacements by

$$-\left(I_3\frac{\mathcal{H}(t)}{\left|\mathcal{H}(t)\right|}\right)\left|\mathcal{H}(l)\right|\mathrm{d}l$$

Let's calculate the result of the right-hand side of (5) in general case when the plane of $I_3\frac{\mathcal{H}(t)}{\left|\mathcal{H}(t)\right|}$ differs from $S(t)$.

To calculate the geometric algebra product of the two exponents in Clifford translation with not coinciding exponent planes, $\mathrm{e}^{I_{S_1}\Delta\varphi_1}\mathrm{e}^{I_{S_2}\varphi_2}$, $S_1\neq S_2$, let's first expand I_{S_1} in original basis to get formulas for generators of Clifford translation. If $I_{S_1}=\gamma_1 B_1+\gamma_2 B_2+\gamma_3 B_3$ then a part of geometrical product $\mathrm{e}^{I_{S_1}\Delta\varphi_1}\mathrm{e}^{I_{S_2}\varphi_2}$ is:

$$\begin{aligned} &I_{S_1}I_{S_2} \\ &=\left(\gamma_1 B_1+\gamma_2 B_2+\gamma_3 B_3\right)\left(\beta_1 B_1+\beta_2 B_2+\beta_3 B_3\right) \\ &=-\left(\gamma_1\beta_1+\gamma_2\beta_2+\gamma_3\beta_3\right)+\left(\gamma_3\beta_2-\gamma_2\beta_3\right)B_1+\left(\gamma_1\beta_3-\gamma_3\beta_1\right)B_2+\left(\gamma_2\beta_1-\gamma_1\beta_2\right)B_3 \\ &=-\left(\boldsymbol{\gamma}\cdot\boldsymbol{\beta}\right)-I_3\left(\boldsymbol{\gamma}\times\boldsymbol{\beta}\right)=I_{S_1}\cdot I_{S_2}+I_{S_1}\wedge I_{S_2} \end{aligned}$$

(see **Figure 3**)

where $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ are vectors dual to bivectors I_{S_1} and I_{S_2}.

Thus, the full product is:

[2]In the case of constant plane of Hamiltonian, it easily follows the Schrodinger equation of conventional quantum mechanics with clearly defined imaginary unit.

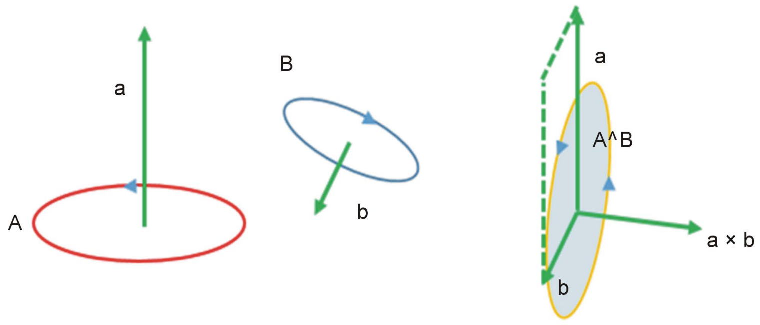

Figure 3. Two bivector geometrical product.

$$\begin{aligned}
&\mathrm{e}^{I_{S_1}\Delta\varphi_1}\mathrm{e}^{I_{S_2}\varphi_2} = \cos\Delta\varphi_1\cos\varphi_2 + \sin\Delta\varphi_1\cos\varphi_2 I_{S_1} \\
&\quad + \cos\Delta\varphi_1\sin\varphi_2 I_{S_2} + \sin\Delta\varphi_1\sin\varphi_2 I_{S_1} I_{S_2} \\
&= \cos\Delta\varphi_1\cos\varphi_2 - \left(\sin\Delta\varphi_1\cos\varphi_2 I_{S_1} I_{S_2}\right) I_{S_2} \\
&\quad - I_{S_1}\left(\cos\Delta\varphi_1\sin\varphi_2 I_{S_1} I_{S_2}\right) + \sin\Delta\varphi_1\sin\varphi_2 I_{S_1} I_{S_2} \\
&= \underbrace{\cos\Delta\varphi_1\cos\varphi_2 + \sin\Delta\varphi_1\sin\varphi_2\left(I_{S_1}\cdot I_{S_2}\right)}_{\text{(scalar part)}} \\
&\quad + \underbrace{\sin\Delta\varphi_1\cos\varphi_2 I_{S_1} + \cos\Delta\varphi_1\sin\varphi_2 I_{S_2} + \sin\Delta\varphi_1\sin\varphi_2 I_{S_1}\wedge I_{S_2}}_{\left(\text{bivector part, expansion in non-orthonormal basis } I_{S_1}, I_{S_2}, I_{S_1}\wedge I_{S_2}\right)}{}^{3}
\end{aligned}$$

4. Transformations of Circular Polarized Electromagnetic Fields

Now we have everything to retrieve action of Clifford translation generated by a Hamiltonian on general solution (4):

$$\mathrm{e}^{-\left(I_3\frac{\mathcal{H}(t)}{|\mathcal{H}(t)|}\right)|\mathcal{H}(t)|\Delta t}\left(\boldsymbol{E}_0 + I_3\boldsymbol{H}_0\right)\left(\lambda\mathrm{e}^{I_S\omega(t-(I_3I_S)\cdot\boldsymbol{r})} + \mu\mathrm{e}^{I_S\omega(t+(I_3I_S)\cdot\boldsymbol{r})}\right)$$

To make expressions simpler I will use notations $I_3\frac{\mathcal{H}(t)}{|\mathcal{H}(t)|} \equiv I_{\mathcal{H}}$, $\omega\left(t-(I_3I_S)\cdot\boldsymbol{r}\right)\equiv\varphi_+$, and $\omega\left(t+(I_3I_S)\cdot\boldsymbol{r}\right)\equiv\varphi_-$. Then we get (see Sections 1.3 and 1.6 in [3] for multiplication details):

$$\begin{aligned}
&\mathrm{e}^{-I_{\mathcal{H}}|\mathcal{H}(t)|\Delta t}\left(\boldsymbol{E}_0 + I_3\boldsymbol{H}_0\right)\left(\lambda\mathrm{e}^{I_S\varphi_+} + \mu\mathrm{e}^{I_S\varphi_-}\right) \\
&= -\left(\boldsymbol{E}_0 + I_3\boldsymbol{H}_0\right)\left(\lambda\mathrm{e}^{-I_{\mathcal{H}}|\mathcal{H}(t)|\Delta t}\mathrm{e}^{I_S\varphi_+} + \mu\mathrm{e}^{-I_{\mathcal{H}}|\mathcal{H}(t)|\Delta t}\mathrm{e}^{I_S\varphi_-}\right) \\
&= -\left(\boldsymbol{E}_0 + I_3\boldsymbol{H}_0\right)\lambda\left(\cos\left(|\mathcal{H}(t)|\Delta t\right)\cos\varphi_+ - \sin\left(|\mathcal{H}(t)|\Delta t\right)\sin\varphi_+\left(I_{\mathcal{H}}\cdot I_S\right)\right) \\
&\quad - \left(\boldsymbol{E}_0 + I_3\boldsymbol{H}_0\right)\lambda\left(\sin\left(|\mathcal{H}(t)|\Delta t\right)\cos\varphi_+ I_{\mathcal{H}} + \cos\left(|\mathcal{H}(t)|\Delta t\right)\sin\varphi_+ I_S\right. \\
&\quad \left. + \sin\left(|\mathcal{H}(t)|\Delta t\right)\sin\varphi_+\left(I_{\mathcal{H}}\wedge I_S\right)\right) - \left(\boldsymbol{E}_0 + I_3\boldsymbol{H}_0\right)\mu\left(\sin\left(|\mathcal{H}(t)|\Delta t\right)\cos\varphi_- I_{\mathcal{H}}\right. \\
&\quad \left. + \cos\left(|\mathcal{H}(t)|\Delta t\right)\sin\varphi_- I_S + \sin\left(|\mathcal{H}(t)|\Delta t\right)\sin\varphi_-\left(I_{\mathcal{H}}\wedge I_S\right)\right)
\end{aligned}$$

[3]In the case $I_{S_1} = I_{S_2}$ we trivially have rotation of $\mathrm{e}^{I_{S_2}\varphi_2}$ by angle $\Delta\varphi_1$.

Let's take popular case of $I_S = B_3 = \hat{x}\hat{y}$ (plane orthogonal to $\hat{z}$ axis) and $I_{\mathcal{H}} = B_1 = \hat{y}\hat{z}$ (or $I_{\mathcal{H}} = B_2 = \hat{z}\hat{x}$, does not matter.) The above formula becomes:

$$
\begin{aligned}
&-(\boldsymbol{E}_0 + I_3\boldsymbol{H}_0)\Big[\lambda\Big(\cos\big(|\mathcal{H}(t)|\Delta t\big)\cos\varphi_+ + \sin\big(|\mathcal{H}(t)|\Delta t\big)\cos\varphi_+ B_1 \\
&+\sin\big(|\mathcal{H}(t)|\Delta t\big)\sin\varphi_+ B_2 + \cos\big(|\mathcal{H}(t)|\Delta t\big)\sin\varphi_+ B_3\Big) \\
&+\mu\Big(\cos\big(|\mathcal{H}(t)|\Delta t\big)\cos\varphi_- + \sin\big(|\mathcal{H}(t)|\Delta t\big)\cos\varphi_- B_1 \\
&+\sin\big(|\mathcal{H}(t)|\Delta t\big)\sin\varphi_- B_2 + \cos\big(|\mathcal{H}(t)|\Delta t\big)\sin\varphi_- B_3\Big)\Big]
\end{aligned}
$$

It makes simpler if F_+ and F_- are equally weighted, say both λ and μ are equal to one:

$$
\begin{aligned}
&-(\boldsymbol{E}_0 + I_3\boldsymbol{H}_0)\Big[\cos\big(|\mathcal{H}(t)|\Delta t\big)(\cos\varphi_+ + \cos\varphi_-) + \sin\big(|\mathcal{H}(t)|\Delta t\big)(\cos\varphi_+ + \cos\varphi_-)B_1 \\
&+\sin\big(|\mathcal{H}(t)|\Delta t\big)(\sin\varphi_+ + \sin\varphi_-)B_2 + \cos\big(|\mathcal{H}(t)|\Delta t\big)(\sin\varphi_+ + \sin\varphi_-)B_3\Big] \\
&= -2(\boldsymbol{E}_0 + I_3\boldsymbol{H}_0)\cos(\hat{z}\cdot\boldsymbol{r})\Big[\cos\big(|\mathcal{H}(t)|\Delta t\big)\cos\omega t + \sin\big(|\mathcal{H}(t)|\Delta t\big)\cos\omega t B_1 \\
&\quad +\sin\big(|\mathcal{H}(t)|\Delta t\big)\sin\omega t B_2 + \cos\big(|\mathcal{H}(t)|\Delta t\big)\sin\omega t B_3\Big)
\end{aligned} \tag{6}
$$

5. Action of Polarization States on Observables

Since a state in the described formalism is operator that gives the result of measurement when acting on observable, which can be any element of geometric algebra G_3, the following is detailed description of the case when the element in parenthesis of the (6) expression acts on some bivector. Such operation is generalization of the Hopf fibration and rotates the bivector in three dimensions.

Denoting[4]:

$$
\begin{aligned}
&\cos\big(|\mathcal{H}(t)|\Delta t\big)\cos\omega t + \sin\big(|\mathcal{H}(t)|\Delta t\big)\cos\omega t B_1 \\
&+\sin\big(|\mathcal{H}(t)|\Delta t\big)\sin\omega t B_2 + \cos\big(|\mathcal{H}(t)|\Delta t\big)\sin\omega t B_3 \\
&\equiv \alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3 \equiv \mathrm{e}^{I_{\mathcal{H},\omega}\psi}
\end{aligned}
$$

where

$$I_{\mathcal{H},\omega} = (\gamma_1 B_1 + \gamma_2 B_2 + \gamma_3 B_3)$$

$$\gamma_1 = \frac{\sin\big(|\mathcal{H}(t)|\Delta t\big)\cos\omega t}{\sqrt{\sin^2\big(|\mathcal{H}(t)|\Delta t\big) + \cos^2\big(|\mathcal{H}(t)|\Delta t\big)\sin^2\omega t}}$$

$$\gamma_2 = \frac{\sin\big(|\mathcal{H}(t)|\Delta t\big)\sin\omega t}{\sqrt{\sin^2\big(|\mathcal{H}(t)|\Delta t\big) + \cos^2\big(|\mathcal{H}(t)|\Delta t\big)\sin^2\omega t}}$$

$$\gamma_3 = \frac{\cos\big(|\mathcal{H}(t)|\Delta t\big)\sin\omega t}{\sqrt{\sin^2\big(|\mathcal{H}(t)|\Delta t\big) + \cos^2\big(|\mathcal{H}(t)|\Delta t\big)\sin^2\omega t}}$$

[4]Easy to see that the left-hand side is unit value element of G_3^+.

$$\psi = \cos^{-1}\left(\cos\left(\left|\mathcal{H}(t)\right|\Delta t\right)\cos\omega t\right)$$

and taking a bivector operand (observable) $c_1B_1 + c_2B_2 + c_3B_3$ we get the result of measurement, action of the state on observable (see [3], [4] for details):

$$\begin{aligned}
&\mathrm{e}^{-I_{\mathcal{H},\omega}\psi}\left(c_1B_1 + c_2B_2 + c_3B_3\right)\mathrm{e}^{I_{\mathcal{H},\omega}\psi} \\
&= \left(c_1\left[\left(\alpha^2 + \beta_1^2\right) - \left(\beta_2^2 + \beta_3^2\right)\right] + 2c_2\left(\beta_1\beta_2 - \alpha\beta_3\right) + 2c_3\left(\beta_1\beta_3 + \alpha\beta_2\right)\right)B_1 \\
&\quad + \left(2c_1\left(\alpha\beta_3 + \beta_1\beta_2\right) + c_2\left[\left(\alpha^2 + \beta_2^2\right) - \left(\beta_1^2 + \beta_3^2\right)\right] + 2c_3\left(\beta_2\beta_3 - \alpha\beta_1\right)\right)B_2 \\
&\quad + \left(2c_1\left(\beta_1\beta_3 - \alpha\beta_2\right) + 2c_2\left(\beta_2\beta_3 + \alpha\beta_1\right) + c_3\left[\left(\alpha^2 + \beta_3^2\right) - \left(\beta_1^2 + \beta_2^2\right)\right]\right)B_3 \\
&= \left(c_1\cos 2\omega t - \sin 2\omega t\left(c_2\cos\left(2\left|\mathcal{H}(t)\right|\Delta t\right) - c_3\sin\left(2\left|\mathcal{H}(t)\right|\Delta t\right)\right)\right)B_1 \\
&\quad + \left(c_1\sin 2\omega t + \cos 2\omega t\left(c_2\cos\left(2\left|\mathcal{H}(t)\right|\Delta t\right) - c_3\sin\left(2\left|\mathcal{H}(t)\right|\Delta t\right)\right)\right)B_2 \\
&\quad + \left(c_2\sin\left(2\left|\mathcal{H}(t)\right|\Delta t\right) + c_3\cos\left(2\left|\mathcal{H}(t)\right|\Delta t\right)\right)B_3
\end{aligned}$$

One interesting remark. If the observable belongs only to the B_1 plane, that's $c_2 = c_3 = 0$, the result of measurement has only components in B_1 and B_2, projections of the value c_1 due to rotation with angular velocity 2ω around the $\hat{z}$ axis.

6. Polarization States Acting on Multiple Observables

The core of quantum computing should not be in entanglement as it understood in conventional quantum mechanics, which only formally follows from representation of the many particle states as tensor products of individual particle states and not supported by really operating physical devices. The core of quantum computing scheme should be in manipulation and transferring of sets of states as operators decomposed in geometrical algebra variant of qubits (g-qubits), or four-dimensional unit sphere points, if you prefer. Such operators can act on observables, particularly through measurements. From the recent calculation we realize that the action of state, which depends on $(t, \boldsymbol{r})$, on an observable can be done only if observable is defined at the same point $(t, \boldsymbol{r})$ where the state is defined [5]. In this way quantum computer is an analog computer keeping information in sets of objects with infinite number of degrees of freedom, contrary to the two value bits or two-dimensional Hilbert space elements, qubits.

Thus, if we have a state

$$\alpha + I_S\beta \equiv \left|g\right\rangle_{(\alpha,\beta,I_S)} = \left|g\right\rangle_{(\alpha(t,\boldsymbol{r}),\beta(t,\boldsymbol{r}),I_S(t,\boldsymbol{r}))}$$

as in the case of polarization defined states, it becomes a state acting on a set of observables if the latter are defined at some given points:

$$\left|c_n\right\rangle = \left|c\right\rangle_{(c(t_n,\boldsymbol{r}_n),I_C(t_n,\boldsymbol{r}_n))} = c_0\left(t_n,\boldsymbol{r}_n\right) + I_{C(t_n,\boldsymbol{r}_n)}\sqrt{1 - c_0^2\left(t_n,\boldsymbol{r}_n\right)},\ n = 1,\cdots,N$$

Then the state transforms into multi-observable one:

Figure 4. Decoding of g-qubit message.

$$\begin{aligned}&\left|g\left(\alpha\left(t_1,\boldsymbol{r}_1\right),\beta\left(t_1,\boldsymbol{r}_1\right),I_S\left(t_1,\boldsymbol{r}_1\right)\right)\cdots g\left(\alpha\left(t_N,\boldsymbol{r}_N\right),\beta\left(t_N,\boldsymbol{r}_N\right),I_S\left(t_N,\boldsymbol{r}_N\right)\right)\right\rangle\\&\equiv\left|g_1\cdots g_N\right\rangle=\iint\left|g\right\rangle_{\left(\alpha(t,\boldsymbol{r}),\beta(t,\boldsymbol{r}),I_S(t,\boldsymbol{r})\right)}\delta\left(\boldsymbol{r}-\boldsymbol{r}_1\right)\delta\left(t-t_1\right)\mathrm{d}\boldsymbol{r}\mathrm{d}t+\cdots\\&\quad+\iint\left|g\right\rangle_{\left(\alpha(t,\boldsymbol{r}),\beta(t,\boldsymbol{r}),I_S(t,\boldsymbol{r})\right)}\delta\left(\boldsymbol{r}-\boldsymbol{r}_N\right)\delta\left(t-t_N\right)\mathrm{d}\boldsymbol{r}\mathrm{d}t\\&=\sum_{n=1}^{n=N}\iint\left|g\right\rangle_{\left(\alpha(t,\boldsymbol{r}),\beta(t,\boldsymbol{r}),I_S(t,\boldsymbol{r})\right)}\delta\left(\boldsymbol{r}-\boldsymbol{r}_n\right)\delta\left(t-t_n\right)\mathrm{d}\boldsymbol{r}\mathrm{d}t\end{aligned}$$

This formula for $\left|g_1\cdots g_N\right\rangle$ bears clear physical and geometrical sense, contrary to conventional quantum mechanics definition following formally from tensor product which does not have good physical interpretation but is the root of entanglement-based quantum computing.

The formula also prompts how quantum encryption decoding can be effectively implemented with the bivector value security key (see Figure 4).

The formula can also be applied to challenging area of anyons in three dimensions.

7. Conclusion

Two seminal ideas—variable and explicitly defined complex plane in three dimensions, and the G_3^+ states[5] as operators acting on observables—allow to put forth comprehensive and much more detailed formalism appropriate for quantum mechanics in general and particularly for quantum computing schemes. The approach may be thought about, for example, as a far going geometric algebra generalization of some proposals for quantum computing formulated in terms of light beam time bins, see [6], [7], but giving much more strength and flexibility in practical implementation.

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[5]Good to remember that "state" and "wave function" are (at least should be) synonyms in conventional quantum mechanics.

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